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Interaction in Scalar Theories of Gravitation

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Abstract

It is shown that O. Bergmann's (1956) scalar field theory is similar to G. Nordström's (1912). The interaction term in the former's theory is equivalent to non-linearising the Nordström theory by including twice the energy density of the field as a source term in the 'Poisson-like' equation. It is further shown that, if the interaction term $(1 + v)$ in Bergmann's theory is replaced by $(1 + v)^2$, then the subsequent field equation appears more reasonable in that the energy density (not twice) appears as a source term.

The first theory of Nordström (1912) was expressed by him as a wave equation for the field, and a gradient law force for the equations of motion. These equations may be derived from the variational principle

$$\delta \int \mathscr{L}\, d^4 x = 0 \tag{1}$$

where the Lagrangean density

$$\mathscr{L} = \tfrac{1}{2}(\Box\varphi)^2 - 4\pi f m_0 \int \exp(\varphi/c^2)\, \delta^4(x - z)\, (\eta_{ij}\dot{z}^i \dot{z}^j)^{1/2}\, ds \tag{2}$$

where η_{ij} is the Minkowskian metric and $\dot{z}^i = dz^i/ds$.

The equation of motion reads

$$\frac{d}{d\tau}\left(m_0 \exp(\varphi/c^2) \frac{dz_i}{d\tau}\right) = m_0 \exp(\varphi/c^2) \frac{\partial\varphi}{\partial z^i} \tag{3}$$

and the field equation reads

$$\Box^2 \varphi = -4\pi f c^2 \gamma \tag{4}$$

where

$$\gamma = \frac{m_0}{c^2} \int \exp(\varphi/c^2)\, \delta^4(x - z)\, (\eta_{ij}\dot{z}^i \dot{z}^j)^{1/2}\, ds$$

or

$$\gamma = m_0 \exp(\varphi/c^2) \left(1 - \frac{v^2}{c^2}\right)^{1/2} \delta^3(x - z(t))$$

where s was chosen to be z^0.

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We might criticise the Nordström theory on the grounds that the gravitational field, as a 'possessor' of energy density, does not appear as a source for itself. The expression given by Nordström for the energy density is

$$\frac{1}{8\pi f}(\Box\varphi)^2 \tag{5}$$

Thus we might lump this together with the density of matter appearing on the right-hand side of (4) so that we would have

$$\Box^2\varphi = -4\pi f c^2\gamma \pm \frac{1}{2c^2}(\Box\varphi)^2 \tag{6}$$

The sign is as yet undetermined. The minus sign suggests an unstable situation wherein matter would be attracted by the field which it creates, so that a body would experience stresses tending to tear it apart all on its own. The plus sign, on the other hand, suggests a picture of matter as a condensation of the field; since now the field appears to be repulsive so that the stresses on bodies are now of the opposite sign of the previous case.

Let us make the change $\psi = (1/c^2)\varphi$ in (6), then this equation reads

$$\Box^2\psi \pm \tfrac{1}{2}(\Box\psi)^2 = -4\pi f\gamma_0\,e^{\psi} \tag{7}$$

where

$$\gamma_0 = m_0\,\delta^3(x - z(t))\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

Now consider the Lagrangean density of Bergmann (1956), which may be written

$$\mathscr{L} = \tfrac{1}{2}(\Box v)^2 - \frac{4\pi f m_0}{c^2}\int (1 + v)\,\delta^4(x - z)\,(\eta_{ij}\dot{z}^i\dot{z}^j)^{1/2}\,ds \tag{8}$$

The field equation reads

$$\begin{aligned}\Box^2 v &= -4\pi f m_0 \int \delta^4(x - z)\,(\dot{z}_i\dot{z}^i)^{1/2}\,ds\\ &= -4\pi f m_0\left(1 - \frac{v^2}{c^2}\right)^{1/2}\delta^3(x - z(t))\end{aligned} \tag{9}$$

and the equation of motion

$$\frac{d}{d\tau}\left[m_0(1 + v)\frac{dz_i}{d\tau}\right] = m_0\frac{\partial v}{\partial z^i} \tag{10}$$

Let us make the substitution

$$1 + v = e^{\psi} \tag{11}$$

The equation of motion now becomes identical with Nordströms. The field equation in terms of ψ then is

$$\Box^2\psi + (\Box\psi)^2 = -4\pi f m_0\exp(-\psi)\left(1 - \frac{v^2}{c^2}\right)^{1/2}\delta^3(x - z(t)) \tag{12}$$

Thus we see that the interaction term is equivalent to the self-coupling of the field in the picture of the force acting on a body as the gradient of the field.

It is interesting to note that, if the interaction term $(1+v)$ in (8) is replaced by $(1+v)^2$ and the substitution (11) is made, there results the field equation

$$\Box^2 \psi + (\Box\psi)^2 = -2\pi f m_0 \left(1 - \frac{v^2}{c^2}\right) \delta^3(x - z(t)) \tag{13}$$

and the equation of motion

$$\frac{d}{d\tau}[m_0 \exp(2\psi)\,\dot{z}_i] = 2m_0 \exp(2\psi)\frac{\partial\psi}{\partial z^i} \tag{14}$$

The substitution $\psi = \frac{1}{2}\Phi$ gives

$$\Box^2 \Phi + \tfrac{1}{2}(\Box\Phi)^2 = -4\pi f m_0 \left(1 - \frac{v^2}{c^2}\right) \delta^3(x - z(t)) \tag{15}$$

and

$$\frac{d}{d\tau}[m_0\, e^{\Phi}\,\dot{z}_i] = m_0\, e^{\Phi}\frac{\partial\Phi}{\partial z^i} \tag{16}$$

The form of equations (15) and (16) suggests that the interaction term should be $(1+v)^2$.

This has a fundamental consequence; for if this interaction in place of $1+v$ appears in (8), then the consistent set of equations (15) and (16) results, in place of (12) and (3). In this last mentioned set, the inertial and active gravitational masses are the same; whereas in (15) and (16) the active gravitational mass is independent of the field. This brings us to one last point, namely what is the field. We have chosen to regard that quantity whose gradient appears in the equation of the motion as the field quantity in spite of the fact that the Langrangean, (3), is expressed in terms of v. There are two good reasons for this, firstly there is a uniform way of comparing the theories by making the equations of motion the same (or similar) and thus comparing field equations and secondly, and perhaps most importantly, we are used to thinking of the gradient of a potential as a force and not the gradient of its logarithm.

References

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